BEHAVIOR OF A LINEAR SYSTEM UNDER SMALL RANDOM EXCITATION OF ITS PARAMETERS

PMM Vol. 31, No. 3, 1967, pp. 527-530

M.B. NEVEL'SON
(Moscow)

(*Received July 20, 1966*)

We consider the behavior of a linear determinate system under the influence of random, Gaussian, white noise fluctuations. In our investigation of the influence of these fluctuations on the stability of the system in question, we shall limit ourselves to the case when the intensity of fluctuations is small. This makes it possible to use the criterion of almost sure stability of a linear stochastic system, given in [1]. In particular, we show that a determinate system unstable in the Liapunov sense remains almost surely unstable on addition of a sufficiently small diffusion term. Analogous statement was put forward as a hypothesis in [2]. A different aspect of the influence of random effects on the stability of the determinate system was considered in [3 and 4].

1. Consider a linear determinate system with constant coefficients

$$X_i^{\cdot} = b_{i1}X_1 + \ldots + b_{in}X_n \qquad (i = 1, \ldots, n) \tag{1.1}$$

For simplicity we shall assume that this system has real and different eigenvalues $\lambda_1, \ldots, \lambda_n$. We can assume without loss of generality that

$$\lambda_1 = \max_i \lambda_i$$

We know that the system (1.1) can be reduced, by means of a nondegenerate linear transformation to a canonical form

$$Z_i^{\cdot} = \lambda_i Z_i \qquad (i = 1, \ldots, n)$$

Let us denote by $|Z|$ the Euclidean norm of vector Z. Then, we have

$$Y = Z / |Z| \tag{1.2}$$

System (1.1) now becomes a dynamic system on a n-dimensional sphere S_n

$$Y_i^{\cdot} = f_i(Y) \qquad (i = 1, \ldots, n) \tag{1.3}$$

Computing a general solution of the system (1.3) we easily see, that the aggregate Γ of stable invariant sets of this system consists of two points: (1, 0,..., 0) and (– 1, 0,..., 0).

2. Assume that the parameters of the system (1.1) are subject to small, random, white noise perturbations. Then Eqs. (1.1) become a system of stochastic differential Eqs.

$$X_i^{\cdot} = \sum_{j=1}^{n} [b_{ij} + \sqrt{\varepsilon}\,\eta_{ij}^{\cdot}(t)]\, X_j \qquad (i = 1, \ldots, n) \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter and $\eta_{ij}^{\cdot}(t)$ are Gaussian white noises with zero mathematical expectation. These need not be independent, hence

$$M\eta_{ik}^{\cdot}(t)\,\eta_{jl}^{\cdot}(s) = 2a_{kl}^{ij}\,\delta(t-s)$$

Solution of (2.1) will be a strictly Markov's, random process $X_\varepsilon(t, x)$ with initial condition $X_\varepsilon(0, x) = x$. We assume that

$$\sum_{i,j,k,l=1}^{n} a_{kl}^{ij} x_k x_l y_i y_j \geqslant c\,|x|^2\,|y|^2, \qquad c > 0$$

i.e. that the process $X_\varepsilon(t, x)$ is nondegenerate.

Passing in the usual manner from the noises $\eta_{ij}^{\cdot}(t)$ to independent white noises, we can write (2.1) as a system of stochastic differential equations of Ito (see e.g. [5]) with a generating operator

$$L = \sum_{i,j=1}^{n} b_{ij} x_j \frac{\partial}{\partial x_i} + \varepsilon \sum_{i,j,k,l=1}^{n} a_{kl}^{ij} x_k x_l \frac{\partial^2}{\partial x_i \partial x_j}$$

We can assume without any loss of generality, that the system (1.1) is already in its canonical form, i.e. that the generating operator of (2.1) (in new coordinates x_i) has the form

$$L_1 = \sum_{i=1}^{n} \lambda_i z_i \frac{\partial}{\partial z_i} + \varepsilon \sum_{i,j=1}^{n} a_{ij}^{(1)}(z) \frac{\partial^2}{\partial z_i \partial z_j} \qquad (2.2)$$

where $a_{ij}^{(1)}(z)$ are quadratic forms in variables z_i. Markov process corresponding to the operator (2.2) will be denoted by $Z_\varepsilon(t, z)$. Obviously, processes $X_\varepsilon(t, x)$ and $Z_\varepsilon(t, x)$ are almost surely simultaneously stable or unstable .

It was shown in [1] that when the transformation (1.2) is applied to $Z_\varepsilon(t, z)$, it becomes a certain random Markov process $Y_\varepsilon(t, y)$ on the sphere S_n. Process $Y_\varepsilon(t, y)$ is described by the differential operator

$$L_2 = \sum_{i=1}^{n} f_i(y) \frac{\partial}{\partial y_i} + \varepsilon \sum_{i,j=1}^{n} a_{ij}^{(2)}(y) \frac{\partial^2}{\partial y_i \partial y_j}$$

Let us put

$$I_\varepsilon = \int_{S_n} A_\varepsilon(y) \mu_\varepsilon(dy), \qquad A_\varepsilon(y) = L_1 \ln |z| = \sum_{i=1}^{n} \lambda_i y_i^2 + \varepsilon A_1(y)$$

where $\mu_\varepsilon(dy)$ is an invariant measure of the process $Y_\varepsilon(t, y)$.

From [1] it follows that if $I_\varepsilon < 0$, then the system (2.1) is almost surely asymptotically stable in the large; if, on the other hand $I_\varepsilon > 0$, then for any $x \neq 0$, we have

$$P\{\lim |X(t, x)| = \infty \quad \text{as} \quad t \to \infty\} = 1 \qquad (2.3)$$

i.e. the system is almost surely unstable.

We shall show in Section 3 that the invariant measure $\mu_\varepsilon(dy)$ of the process $Y_\varepsilon(t, y)$ converges, as $\varepsilon \to 0$, to some invariant measure $\mu_0(dy)$ of the limit dynamic system (1.3). In addition, the measure $\mu_0(dy)$ is wholly concentrated on the aggregate Γ of the sphere S_n, Γ containing all stable invariant sets of the system (1.3).

From the above and from the form of Γ, we have

$$I_\varepsilon = \lambda_1 \mu_\varepsilon(\Gamma) + \varepsilon \int_\Gamma A_1(y) \mu_\varepsilon(dy) + \int_{S_n \setminus \Gamma} A_\varepsilon(y) \mu_\varepsilon(dy) = \lambda_1 + \alpha_1(\varepsilon), \qquad (2.4)$$

$$\lim \alpha_1(\varepsilon) = 0 \quad \text{as} \quad \varepsilon \to 0$$

Proof of (2.4) is conducted under the assumption that the eigenvalues λ_i $(i = 1,\ldots,n)$ are real and different. A relation analogous to (2.4) can also easily be obtained for a general case when (1.1) has arbitrary eigenvalues, by reducing (1.3) to a canonical form and considering a resulting dynamic system on the sphere S_n. Investigating the aggregate Γ of stable invariant sets of this system(*) we can similarly show that

$$I_\varepsilon = \lambda + \alpha(\varepsilon), \qquad \lambda = \max_i \operatorname{Re} \lambda_i, \qquad \lim \alpha(\varepsilon) = 0, \quad \text{as} \quad \varepsilon \to 0 \qquad (2.5)$$

The following facts emerge from Eqs. (2.5).

1) If the system (1.1) is unstable, i.e. if $\lambda > 0$, then a sufficiently small ε_0 can be found such that (2.1) will also almost surely be unstable in the sense of (2.3) for all $\varepsilon < \varepsilon_0$.

2) If the system (1.1) is asymptotically stable, i.e. $\lambda < 0$, the (2.1) will also be almost surely asymptotically stable in the large for all sufficiently small ε (see [2 and 7]).

3) If the system (1.1) is stable but not asymptotically, i.e. if $\lambda = 0$, then additional terms of asymptotic expansion of I_ε as $\varepsilon \to 0$ should be obtained, before the problem of stability of (2.1) can be decided. The general method of obtaining such an expression is fairly complicated, but results of [8] make it possible to obtain it easily for $n = 2$. We also find that

---

*) We should note that linearly independent solutions of (1.1) corresponding to a root λ_i can, in the general case, be divided into a definite number of groups of solutions [6]. Γ will then depend on the number of solutions in the groups corresponding to characteristic roots with a largest real part.

if the second eigenvalue of (1.1) is negative, then (2.1) is almost surely asymptotically stable in the large for all sufficiently small ε; if on the other hand both eigenvalues are purely imaginary, then (2.1) can be either stable, or almost surely unstable, depending on the diffusion coefficients a_{kl}^{ij}.

3. To conclude the proof of above statements in Section 2, it is sufficient to establish the following general fact(*): the invariant measure $\mu_\varepsilon(dy)$ of a nondegenerate Markov random process $Y_\varepsilon = \{Y_\varepsilon(t, y), P_\varepsilon\}$ given on a sphere and defined by the operator(**)

$$L = \sum_{i=1}^{n} b_i(y) \frac{\partial}{\partial y_i} + \varepsilon \sum_{i,j=1}^{n} a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j}$$

converges, as $\varepsilon \to 0$, to an invariant measure $\mu_0(dy)$ of a limiting dynamic system

$$Y_i^{\cdot} = b_i(Y) \qquad (i = 1, \ldots, n) \tag{3.1}$$

and the measure $\mu_0(dy)$ is wholly concentrated on stable invariant sets of the system (3.1).

Let Γ be an aggregate of stable invariant sets of (3.1) and let Γ_δ be a δ-neighborhood of Γ. Assume in addition, that K is an aggregate of unstable invariant sets of the system (3.1) and K_γ is a γ-neighborhood of K where δ and γ are so small, that $K_\gamma \cap \Gamma_\delta = \varnothing$. Then, for any $y \in A_\gamma = S_n \setminus K_\gamma$ we can find such $t_0 = t_0(\delta, \gamma)$ that the trajectory $Y_0(t, y)$ of the dynamic system (3.1) originating at the initial moment at the point y, belongs to the set $\Gamma_{1/2\delta}$ at all $t \geqslant t_0$. This, together with the asymptotic expansion for diffusion processes dependent on small parameter [10], infers that at any fixed $t \geqslant t_0$, for $y \in A_\gamma$,

$$\lim P_\varepsilon\{Y_\varepsilon(t, y) \in \Gamma_\delta\} = 1 \quad \text{as} \quad \varepsilon \to 0 \tag{3.2}$$

uniformly in $y \in A_\gamma$.

Let us now denote by $\tau_\varepsilon(y)$ the instant at which the trajectory of $Y_\varepsilon(t, y)$ leaves the set K_γ for the first time. Proof of Theorem 4 in [11] easily yields the following relation valid for some $T > 0$ which states, that

$$\lim P_\varepsilon\left\{\tau_\varepsilon(y) \geqslant \frac{T}{\varepsilon^2}\right\} = 0 \quad \text{as} \quad \varepsilon \to 0 \tag{3.3}$$

uniformly in $y \in K_\gamma$.

Using (3.2), (3.3) and a specific Markov's property we can easily show that as $\varepsilon \to 0$,

$$P_\varepsilon\left\{y, \frac{T+1}{\varepsilon^2}, K_\gamma\right\} = P_\varepsilon\left\{Y_\varepsilon\left(\frac{T+1}{\varepsilon^2}, y\right) \in K_\gamma\right\} \to 0 \tag{3.4}$$

uniformly in $y \in S_n$.

Moreover, from the definition of invariant measure $\mu_\varepsilon(dy)$ we have for any open set on the sphere S_n and any $t > 0$,

$$\mu_\varepsilon(U) = \int_{S_n} P_\varepsilon\{y, t, U\} \mu_\varepsilon(dy) \tag{3.5}$$

Eqs. (3.4) and (3.5) together, yield (similar method was used in [9])

$$\lim \mu_\varepsilon(K_\gamma) = 0 \quad \text{as} \quad \varepsilon \to 0 \tag{3.6}$$

Consequently from (3.2), (3.5) and (3.6), we obtain

$$\lim \mu_\varepsilon(\Gamma_\delta) = 1 \quad \text{as} \quad \varepsilon \to 0$$

Last equation which is valid for all sufficiently small δ, implies that the invariant measure $\mu_\varepsilon(dy)$ converges, as $\varepsilon \to 0$, to a certain measure $\mu_0(dy)$ wholly concentrated on Γ.

The author thanks R.Z. Khas'minskii for valuable suggestions

BIBLIOGRAPHY

1. Khas'minskii, R.Z., Sufficient and necessary conditions of almost surely asymptotic

*) The problem of behavior of an invariant measure of a Markov random process with a weak diffusion was investigated in [9].

**) We assume that the transport vector and the diffusion matrix of the process Y_ε satisfy Lipshits conditions.

stability of a linear stochastic system. Theory of probability and its application, (Russian Journal) Vol. 11, No. 1, 1966.
2. Khas'minskii, R.Z., On the stability of the trajectory of Markov processes. PMM, Vol. 26, No. 6, 1962.
3. Leibowitz, A.M., Statistical behavior of linear systems with random varying parameters. J. Math. end Phys., Vol. 6, No.6, 1963.
4. Rabotnikov, Iu.L., On the impossibility of stabilizing a system in the mean-square by random perturbation of its parameters. PMM, Vol. 28, No. 5, 1964.
5. Doob, J.L., Stochastic processes, J. Wiley and Sons, 1953, (Russian translation 1956).
6. Malkin, I.G., Some Problems in the Theory of Nonlinear Oscillations. M., Gostekhizdat, 1956.
7. Khas'minskii, R.Z., Equations with random perturbations . Theory of probability and its application (Russian Journal), Vol. 10, No. 2, 1965.
8. Nevel'son, M.B., Behavior of the invariant measure of a diffusion process with low diffusion at the boundary. Theory of probability and its application (Russian Journal), Vol. 9, No. 1, 1964.
9. Khas'minskii, R.Z., On the averaging principle for parabolic and elliptic differential equations and Markov processes with low diffusion. Theory of probability and its application (Russian Journal), Vol. 9, No. 1, 1963.
10. Blagoveshchenskii, Iu.N., Diffusion processes dependent on small parameters. Theory of probability and its application (Russian Journal) Vol. 7, No. 1, 1962.
11. Freidlin, M.I., On Ito stochastic equations and degenerate elliptic equations. Izv. Akad. Nauk SSSR, Ser. mat., Vol. 26, No. 5, 1962.

Translated by L.K.